

Which operator generates time evolution in Quantum Mechanics?

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Abstract

We attract attention to an interesting family of quantum systems where the generator $H_{(gen)}$ of time-evolution of wave functions is *not* equal to the Hamiltonian H . We describe the origin of the difference $H_{(gen)} - H$ and interpret it as a carrier of a compressed information about the other relevant observables.

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In the great majority of the reviews of the history of Quantum Mechanics (cf., e.g., ref. [1] for illustration) the main emphasis is usually being laid upon its successful descriptions of the various *not too complicated* systems. The admissible states Φ are then often just bound states which lie in the standard Hilbert space $\mathcal{H}_{phys}^{(stand)}$ (in what follows we shall write its elements in the specific format $|\Phi\rangle$ using the slightly deformed Dirac's kets). Similarly, the Hamiltonians (denoted by the lower-case h) and/or the other observables are usually chosen as some sufficiently elementary (say, differential) operators in $\mathcal{H}_{phys}^{(stand)}$.

In such a setting, one can often make use of the knowledge of the spectral representation of the Hamiltonian

$$h = \sum_{n=0}^{\infty} |n\rangle E_n \langle n| \quad (1)$$

etc. A breakdown of the idyllic situation is encountered, say, in the various many-fermion problems where a constructive treatment of the action of a realistic h on a generic $|\Phi\rangle \in \mathcal{H}_{phys}^{(stand)}$ may prove overcomplicated and, for various more or less purely technical reason, even prohibitively difficult. In such a context and in a way known and recommended, say, in nuclear physics [2], people usually search for a suitable mapping of the complicated h on some “equivalent” but simpler operator H . In general, the latter operator would act in some other, auxiliary, “reference” Hilbert space $\mathcal{H}^{(ref)}$. Typically [2], a “realistic” multinucleon Hamiltonian h acting in $\mathcal{H}_{phys}^{(stand)}$ can be assigned an idealized isospectral bosonic partner H acting in some other Hilbert space $\mathcal{H}^{(ref)}$.

At present, we may already read about a quickly growing number of applications of the above idea in several other branches of quantum theory (cf., e.g., the recent Carl Bender's thorough review [3] of the promising features of the so called \mathcal{PT} -symmetric models in field theory, etc). The use of several Hilbert spaces clarified also several older puzzles. The list of successes ranges from the suppression of the negative Klein-Gordon probabilities [4] up to the “legalization” of the spurious states in the Lee model [5]. In all of these settings, a decision of working with the mappings of the Hamiltonians,

$$h = \Omega H \Omega^{-1} \quad (2)$$

relies on the key tacit assumption that the simplicity of the calculations with the transformed H will more than compensate for some complications arising due to the presence of the mapping Ω .

A decisive breakthrough in the appreciation of the merits of the use of several Hilbert spaces at once came with the Bender's and Boettcher's discovery [6] that the reading of the above eq. (2) can be also inverted. Thus, an innovated model-building in quantum theory could equally well start from the choice of some mathematically tractable model H which would be followed by a suitable transition to its “intractable” but “physical” equivalent partner h . Of course, all the similar applications must be based on an appropriate and consistent quantum-theoretical background. Some of its less usual aspects will be discussed in what follows.

Firstly, a suitable operator Ω should be introduced and understood as a transformer of h into

H and *vice versa*. With the left-hand side of eq. (2) considered as acting on a given ket vector $|\Phi\rangle \in \mathcal{H}_{phys}^{(stand)}$ we have to perceive the modified upper-case operator H as acting on a ket-vector element $|\Phi\rangle$ of *another*, reference space $\mathcal{H} = \mathcal{H}^{(ref)}$. Once we use the standard Dirac's bras and kets we get, up to inessential multiplication constants,

$$|\Phi\rangle := \Omega^{-1} |\Phi\rangle \in \mathcal{H}^{(ref)}, \quad \langle\Phi| := \langle\Phi| \Omega^{-1} \dagger \in [\mathcal{H}^{(ref)}]^\dagger \sim \mathcal{H}^{(ref)}. \quad (3)$$

It is worth noticing that by definition, *both* the old and new spaces are *self-dual*. At the same time they are *not* unitary equivalent since, by construction,

$$\langle\Phi|\Phi'\rangle = \langle\Phi|\Omega^\dagger\Omega|\Phi'\rangle. \quad (4)$$

We see that still another Hilbert space has to be introduced. Let us denote it by the symbol \mathcal{H}_{phys} without superscript. It will *share* its ket vectors with the “intermediate” space $\mathcal{H} = \mathcal{H}^{(ref)}$ (where the inner product was $\langle\cdot|\cdot\rangle$). In parallel, it will *differ* from it by the innovated definition of its linear functionals,

$$\mathcal{H}_{phys}^\dagger := \{ \langle\Phi| := \langle\Phi|\Omega^\dagger\Omega \equiv \langle\Phi|\Omega \} . \quad (5)$$

In this sense, only the mapping between $\mathcal{H}_{phys}^{(stand)}$ and \mathcal{H}_{phys} can be considered norm-preserving and unitary.

Once we assume that H acting in \mathcal{H}_{phys} is our “simplest” (i.e., e.g., ordinary differential) operator, we may combine eq. (2) with conventions (3) and (5) and write down the spectral representation of H ,

$$H = \sum_{n=0}^{\infty} |n\rangle E_n \langle n|. \quad (6)$$

In an opposite direction, the appropriately normalized biorthogonal basis used in (6) can be defined as composed of the doublets of the left and right eigenvectors of H ,

$$H |n\rangle = E_n |n\rangle, \quad \langle n| H = \langle n| E_n, \quad \langle m|n\rangle = \delta_{mn}, \quad I = \sum_{n=0}^{\infty} |n\rangle \langle n|.$$

Via insertion of eq. (1) in eq. (2) it is now easy to deduce that we may have started from the whole family of the maps Ω defined by the (presumably, convergent) infinite series

$$\Omega = \sum_{n=0}^{\infty} |n\rangle \mu_n \langle n|. \quad (7)$$

Here *all* the constants $\mu_n \in \mathbb{C} \setminus \{0\}$ are free parameters. Once we know that $h = h^\dagger$, i.e.,

$$\Omega H \Omega^{-1} = [\Omega^{-1}]^\dagger H^\dagger \Omega^\dagger \quad (8)$$

we immediately deduce that

$$H^\dagger = \Theta H \Theta^{-1} \quad (9)$$

where we abbreviated

$$\Theta = \Omega^\dagger \Omega = \sum_{m=0}^{\infty} |m\rangle\langle m| \mu_m^2 \langle\langle m| = \Theta^\dagger. \quad (10)$$

The latter formula offers a formal key to the consistent work in the Hilbert space $\mathcal{H}_{phys} \equiv \mathcal{H}^{(\Theta)}$ since *any* mean value of an operator \mathcal{O} of an observable must be real,

$$\langle a | \Theta \mathcal{O} | a \rangle = \langle a | \mathcal{O}^\dagger \Theta | a \rangle$$

so that we have to demand that

$$\mathcal{O}^\dagger = \Theta \mathcal{O} \Theta^{-1}. \quad (11)$$

The scope of this general framework of quantum theory can be illustrated by the Bessis' toy non-Hermitian Hamiltonian

$$H(g) = p^2 + i g x^3, \quad g > 0. \quad (12)$$

It possesses the real and discrete spectrum of energies which is bounded from below [6, 7]. Moreover, in spite of its manifest non-Hermiticity in $\mathcal{H}^{(ref)} = L_2(\mathbb{R})$, it has been shown [8] essentially self-adjoint (and, hence, “fully legal”) in the Hilbert space $\mathcal{H}^{(\Theta)}$ which differs from $L_2(\mathbb{R})$ *solely* by *another definition* of the inner product between elements $|a\rangle$ and $|b\rangle$,

$$(a, b)^{(\Theta)} = \langle a | \Theta | b \rangle. \quad (13)$$

Such an introduction of the new space may be treated as a mere transition to an alternative metric operator $\Theta = \Theta^\dagger > 0$ in the old space.

A certain climax of our present letter comes when we address the problem of the time-evolution in Quantum Mechanics with $\Theta \neq I$ and with a manifest time-dependence in all our operators. Once we prepare an initial state as a normalized vector $|\varphi(t)\rangle \in \mathcal{H}_{phys}^{(stand)}$ at $t = 0$, we can rely only on our understanding of the evolution caused by the auxiliary self-adjoint Hamiltonians $h(t)$. In particular, we may immediately solve any time-dependent Schrödinger equation

$$i \partial_t |\varphi(t)\rangle = h(t) |\varphi(t)\rangle, \quad |\varphi(t)\rangle = u(t) |\varphi(0)\rangle \quad (14)$$

and we are sure that the related evolution operator given by equation

$$i \partial_t u(t) = h(t) u(t) \quad (15)$$

is *certainly* unitary in $\mathcal{H}_{phys}^{(stand)}$,

$$\langle \varphi(t) | \varphi(t) \rangle = \langle \varphi(0) | \varphi(0) \rangle.$$

In the next step of our considerations we recollect the pull-backs $|\Phi(t)\rangle = \Omega^{-1}(t) |\varphi(t)\rangle$ and $\langle\langle \Phi(t) | = \langle \varphi(t) | \Omega(t)$ carrying, by assumption, their own time dependence. It is represented, formally, by the “right-action” evolution rule

$$|\Phi(t)\rangle = U_R(t) |\Phi(0)\rangle, \quad U_R(t) = \Omega^{-1}(t) u(t) \Omega(0) \quad (16)$$

accompanied by its “left-action” parallel

$$|\Phi(t)\rangle\rangle = U_L^\dagger(t) |\Phi(0)\rangle\rangle, \quad U_L^\dagger(t) = \Omega^\dagger(t) u(t) [\Omega^{-1}(0)]^\dagger. \quad (17)$$

The respective non-Hermitian analogues of the Hermitian evolution rule (15) are now obtained by the elementary differentiation and insertions yielding the two separate differential operator equations

$$i\partial_t U_R(t) = -\Omega^{-1}(t) [i\partial_t \Omega(t)] U_R(t) + H(t) U_R(t) \quad (18)$$

and

$$i\partial_t U_L^\dagger(t) = H^\dagger(t) U_L^\dagger(t) + [i\partial_t \Omega^\dagger(t)] [\Omega^{-1}(t)]^\dagger U_L^\dagger(t). \quad (19)$$

We are prepared to verify what happens with the norm $\langle\langle \Phi(t) | \Phi(t) \rangle\rangle$ of states which evolve with time in the physical space $\mathcal{H}^{(\Theta)}$. The elementary differentiation gives

$$i\partial_t \langle\langle \Phi(t) | \Phi(t) \rangle\rangle = i\partial_t \langle\langle \Phi(0) | U_L(t) U_R(t) | \Phi(0) \rangle\rangle = \quad (20)$$

$$= \langle\langle \Phi(0) | [i\partial_t U_L(t)] U_R(t) | \Phi(0) \rangle\rangle + \langle\langle \Phi(0) | U_L(t) [i\partial_t U_R(t)] | \Phi(0) \rangle\rangle = \quad (21)$$

$$= \langle\langle \Phi(0) | U_L(t) [-H(t) + \Omega^{-1}(t) [i\partial_t \Omega(t)]] U_R(t) | \Phi(0) \rangle\rangle + \quad (22)$$

$$+ \langle\langle \Phi(0) | U_L(t) [H(t) - \Omega^{-1}(t) [i\partial_t \Omega(t)]] U_R(t) | \Phi(0) \rangle\rangle = 0. \quad (23)$$

We see that the norm remains constant also in $\mathcal{H}^{(\Theta)}$. In the other words, the time-evolution of the system specified by the quasi-Hermitian and time-dependent Hamiltonian $H(t)$ is unitary.

Once we abbreviate $\dot{\Omega}(t) \equiv \partial_t \Omega(t)$, our latter observation can be rephrased as an explicit specification of *the same* time-evolution generator

$$H_{(gen)}(t) = H(t) - i\Omega^{-1}(t)\dot{\Omega}(t) \quad (24)$$

entering *both* the present quasi-Hermitian updates

$$i\partial_t |\Phi(t)\rangle = H_{(gen)}(t) |\Phi(t)\rangle, \quad (25)$$

$$i\partial_t |\Phi(t)\rangle\rangle = H_{(gen)}^\dagger(t) |\Phi(t)\rangle\rangle \quad (26)$$

of the current textbook time-dependent Schrödinger equation for wave functions. Such a confirmation of the preservation of the overall unitarity of the evolution (at the cost of a not too difficult and explicit redefinition $H \rightarrow H_{(gen)}$ of the generator of time evolution which remains the same for both the left and right action) may be read as really good news for the theory.

We arrived at the answer to the question given in the title. In a concise discussion we should emphasize that such an answer is quite surprising because the standard connection between the Hamiltonian and the time evolution of the system is usually interpreted as a certain “first principle” of the quantum dynamics. In such a context our present constructive argument against the current postulate $H_{(gen)} = H$ should be perceived as a formal foundation of an apparently counterintuitive innovative idea that once we admit some dynamically motivated time dependence of

the Hamiltonian itself, we might also contemplate a parallel assumption of some equally arbitrary, phenomenologically motivated time dependence of some other observable quantities.

The key reason for the possible mathematical consistency as well as for a formal acceptability of such a fairly unusual scenario should be seen in the deep formal ambiguity of the metric operator. In principle, this observation (made also in refs. [2] and [9]) gives us a huge space and freedom for the *time-dependent* variability of the metric $\Theta = \Theta(t)$. Let us emphasize that in the generic case with $\dot{H}(t) \neq 0$ the t -dependence of $H \neq H^\dagger$ becomes immediately transferred to its left and right eigenvectors and, subsequently (i.e., via formula (10)), to the metric $\Theta = \Theta(t)$ itself. Moreover, in parallel to the freedom of our choice of the t -dependence of the Hamiltonian $H(t)$, there exists, at least in principle, no mathematical obstruction to an unrestricted time-dependence freedom introduced directly in all the complex coefficients $\mu_n = \mu_n(t)$ (cf., once more, eq. (10)).

We have to admit that we did not really expect that the “old” idea of a mapping Ω between spaces [2] (and, of course, between Hamiltonian H and its “suitable” Hermitian partner h) will survive so easily its transfer to the time-dependent case with $\Omega = \Omega(t)$. At the same time, having arrived at our final pair of the generalized quantum time-evolution differential equations (25) and (26) we find them quite natural and consistent. Summarizing our present results, the Hamiltonian (i.e., energy-operator $H = H(t)$) was assumed quasi-Hermitian (i.e., $H^\dagger(t) = \Theta(t) H(t) \Theta^{-1}(t)$ at some nontrivial metric $\Theta(t) = \Theta^\dagger(t) > 0$). In parallel, some other quasi-Hermitian operators A_j of observables were assumed to carry an independent time-dependence (i.e., we were allowed to demand that $A_j^\dagger(t) = \Theta(t) A_j(t) \Theta^{-1}(t)$). Under these assumptions we demonstrated that the evolution of the system in question still remains unitary.

The latter observation throws new light upon some applications of the models with $\Theta \neq I$ (cf., e.g., the ones reviewed in [3]) where the values of all the coefficients $\mu_n = \mu_n(t)$ are fixed using a suitable phenomenological postulate. In the other applications reviewed in [2] the variability of μ_n s is also being removed, step by step, via an appropriate selection of some other operators decided to play the role of observables. We may conclude that there is nothing counterintuitive in the manifest “decoupling” of $H_{(gen)}(t)$ from $H(t)$. It is fairly easily accepted immediately after one imagines that the information about the quantum evolution *can* be carried *not only* by the manifestly time-dependent Hamiltonian $H(t)$ *but also* by some other manifestly time-dependent observables $A_j(t)$, $j = 1, 2, \dots$. Naturally, the latter information must also influence the overall evolution of the system.

In each of the eligible scenarios and approaches, the constraints imposed upon the variability of the coefficients in $\Omega(t)$ may be considered controllable and at our disposal. After an acceptance of such a concept of the time-dependence in quantum dynamics, our present main *technical* contribution may be seen in an almost elementary check of the mutual compatibility and independence of the two “input-information” hypotheses involving $H = H(t)$ and $\Theta = \Theta(t)$.

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